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THEORETICAL ANALYSIS OF A PULSE TUBE REGENERATOR

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ABSTRACT

A theoretical analysis of the behavior of a typical pulse tube regenerator has been carried out. Assuming simple sinusoidal oscillations, the static and oscillatory pressures, velocities and temperatures have been determined for a model that includes a compressible gas and imperfect thermal contact between the gas and the regenerator matrix. For realistic material parameters, the analysis reveals that the pressure and velocity oscillations are largely independent of details of the thermal contact between the gas and the solid matrix. Only the temperature oscillations depend on this contact. Suggestions for optimizing the design of a regenerator are given.

CONSERVATION EQUATIONS

The conservation equations for an element of gas undergoing three-dimensional flow and exchanging heat with its surroundings, ignoring gravitational forces, can be written as follows[1] (we adopt the notation that variables with an asterisk are the normal unscaled variables; later we will introduce scaled, dimensionless variables, which will not have an asterisk):

Mass conservation:
$$\frac{\partial \rho^*}{\partial t^*} = -\nabla \cdot \rho^* \mathbf{v}^* \qquad (1)$$

Momentum conservation:
$$\frac{\partial \rho^* \mathbf{v}^*}{\partial t^*} = -\nabla \cdot \rho^* \mathbf{v}^* \mathbf{v}^* - \nabla P^* - \nabla \cdot \tau^* \qquad (2)$$

Energy conservation:
$$\frac{\partial}{\partial t^*}(\rho^* U^*) = -\nabla \cdot \rho^* U^* \mathbf{v}^* - P^* \nabla \cdot \mathbf{v}^* - \nabla \cdot \mathbf{q}^* - \tau^* : \nabla \mathbf{v}^* \qquad (3)$$

where **v*** is the vector velocity, τ***** is the stress tensor, **q*** is heat flow, and U* is the internal energy density.

In the case of one-dimensional flow in the z-direction, eqs. (1), (2) and (3) become (using eq. (1') to arrive at the form of eqs. (2') and (3')), in cylindrical coordinates:

$$\frac{\partial \square^*}{\partial t^*} = -\frac{\partial(\square^* v_z^*)}{\partial z^*} \tag{1'}$$

$$\square^* \frac{\partial v_z^*}{\partial t^*} = -\square^* v_z^* \frac{\partial v_z^*}{\partial z^*} - \frac{\partial P^*}{\partial z^*} + \square\left(\frac{\partial^2 v_z^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial v_z^*}{\partial r^*} + \frac{4}{3}\frac{\partial^2 v_z^*}{\partial z^{*2}}\right) \tag{2'}$$

$$\square^* C_p^* \frac{\partial T^*}{\partial t^*} - \frac{\partial P^*}{\partial t^*} = -\square^* v_z^* C_p \frac{\partial T^*}{\partial z^*} - k\frac{\partial^2 T^*}{\partial z^{*2}}$$

$$+ v_z^* \frac{\partial P^*}{\partial z^*} + Q + \square\left[\left(\frac{\partial v_z^*}{\partial r^*}\right)^2 + \frac{4}{3}\left(\frac{\partial v_z^*}{\partial z^*}\right)^2\right] \tag{3'}$$

where U^* has been replaced by $C_p T^* - P^*/\square^*$ and the $\square \cdot \mathbf{q}^*$ term has been split up into an axial conduction term, $\frac{\partial}{\partial z^*}(k \frac{\partial T^*}{\partial z^*})$, and a term, Q, for heat transfer from the matrix. In eqs. (2') and (3') the stress tensor, $\square^*$, has been expressed in terms of its velocity gradient components [1].

The terms involving $\partial v_z^*/\partial z^*$ in eqs. (2') and (3') can be neglected and in most regenerator calculations the term $\square^* \partial v_z^*/\partial t^*$ in eq. (2') is also very small compared to the pressure gradient and the viscous terms. In that case eq. (2') can be used to evaluate the term, $v_z^* \partial v_z^*/\partial t^*$, in eq. (3') and one obtains [2]:

$$\square^* C_p \frac{\partial T^*}{\partial t^*} - \frac{\partial P^*}{\partial t^*} = -\square^* \frac{q^*}{f} C_p \frac{\partial T^*}{\partial z^*} + k\frac{\partial^2 T^*}{\partial z^{*2}} + Q \tag{4}$$

where v_z^* has been replaced by $q^*/f = u^*_{av}$ by averaging v_z^* over the cross section of the pore; q^* is the effective velocity that the fluid would have with no matrix present and f is the void fraction.

The $\partial v_z^*/\partial r^*$ terms in eq. (2') are generally replaced by a Darcy permeability expression for a porous matrix. Then eq. (2') becomes:

$$\frac{\square^*}{f}\frac{\partial q^*}{\partial t^*} = -\frac{\partial P^*}{\partial z^*} - \frac{\square q^*}{K_p} \tag{5}$$

where K_p is the Darcy permeability.

To convert the variables to dimensionless form we use the following scaling parameters: P_0^*, average pressure in the system, T_0^*, temperature at hot end of regenerator, $\square_0^*$, density at hot end of regenerator, q_0^*, gas velocity at hot end of regenerator, L_0^*, length of regenerator and $\square^*$, period of oscillation. These scaling parameters can be combined into dimensionless parameters as follows:

$\square = \square^* q_0^*/L_0^*$; $\square = C_p/C_v$, M = Mach number = $q_0^*/\sqrt{\square R T_0^*}$, where R is the gas constant, Va = Valensi number = $K_p \square_0^*/\square\,\square^*$; $\square = f/2\square\,\mathrm{Va}$, where f is the void fraction. These parameters are convenient when expressing the previous equations in dimensionless form:

Scaled mass conservation equation, eq. (1'):

$$\frac{\partial \square}{\partial t} + \frac{\square}{f}\frac{\partial}{\partial z}(\square q) = 0 \tag{6}$$

Scaled ideal gas law:

$$P = \square T \tag{7}$$

Scaled momentum conservation equation, eq. (5):

$$\frac{\square}{f}\frac{\partial q}{\partial t} = -\frac{\square}{\square M^2}\frac{\partial P}{\partial z} - \frac{q}{\mathrm{Va}} \tag{8}$$

Scaled energy conservation equation, eq. (4):

$$\square_0^* C_p\left(\frac{\partial T}{\partial t}+\frac{\square q}{f}\frac{\partial T}{\partial z}\right)\square - R\,\square_0^*\frac{\partial P}{\partial t}=\frac{\square^* k}{L_0^{*2}}\frac{\partial^2 T}{\partial z^2}+\frac{\square^*}{T_0^*}Q \qquad (9)$$

The heat transferred between the gas and the matrix (per unit volume of gas) is: $Q = h\,A_s(T_m^* - T^*)/V_{gas}$ where h is the heat transfer coefficient, A_s is the area of the gas–matrix interface and V_{gas} is the gas fraction of the total volume. The hydraulic radius is $r_h = V_{gas}/A_s$ so $Q = h\,(T_m^* - T^*)/r_h = h\,T_0^*(T_m - T)/r_h$ in dimensionless variables. Therefore, eq. (9) becomes:

$$\square_0^* C_p\left(\frac{\partial T}{\partial t}+\frac{\square q}{f}\frac{\partial T}{\partial z}\right)\square - R\,\square_0^*\frac{\partial P}{\partial t}=\frac{\square^* k}{L_0^{*2}}\frac{\partial^2 T}{\partial z^2}+\square^*\frac{h}{r_h}\left(T_m - T\right) \qquad (10)$$

A similar energy conservation equation can be written for the matrix:

$$\square_m^* C_m (1-f)\left(\frac{\partial T_m}{\partial t}\right)=\frac{\square^* k_m(1-f)}{L_0^{*2}}\frac{\partial^2 T_m}{\partial z^2}+\square^* f\,\frac{h}{r_h}\left(T - T_m\right) \qquad (11)$$

where $\square_m$ and C_m are the density and heat capacity of the matrix, T_m is the scaled temperature of the matrix and k_m is the thermal conductivity of the matrix. The term describing the heat flow from the gas (per gas volume) has been multiplied by f, the gas fraction of the total volume, and the terms involving just the matrix are multiplied by (1 – f), the matrix fraction of the total volume.

SOLUTION OF EQUATIONS

These equations will be solved in a perturbation treatment that is valid when the level of oscillation of the gas is small. The perturbation parameter is $\square$ which is proportional to q_0^*, the amplitude of the velocity oscillation at the hot end of the regenerator. Further, it is assumed that all the important features of the system operation can be described with only fundamental frequency terms and that higher harmonic terms can be neglected. These assumptions lead to the following expansions for the dimensionless variables in the above equations:

$$\begin{aligned}
P &= P_0 + \square\left(P_a + P_d\,e^{i2\square t}\right) + \square^2\left(P_{2a} + P_{2d}\,e^{i2\square t}\right)\\
T &= T_0 + \square\left(T_a + T_d\,e^{i2\square t}\right) + \square^2\left(T_{2a} + T_{2d}\,e^{i2\square t}\right)\\
T_m &= T_0 + \square\left(T_{ma} + T_{md}\,e^{i2\square t}\right) + \square^2\left(T_{m2a} + T_{m2d}\,e^{i2\square t}\right)\\
\square &= \square_0 + \square\left(\square_a + \square_d\,e^{i2\square t}\right) + \square^2\left(\square_{2a} + \square_{2d}\,e^{i2\square t}\right)\\
q &= q_a + q_d\,e^{i2\square t} + \square\left(q_{2a} + q_{2d}\,e^{i2\square t}\right)
\end{aligned} \qquad (12)$$

Subscripts a refer to average values and subscripts d refer to the amplitude of oscillating or dynamic variables.

When eqs. (12) are substituted into eqs. (6), (7), (8), (10) and (11) the lowest order equations describing the oscillating-term coefficients are:

$$\frac{1}{f}\frac{\partial(\square_0\,q_d)}{\partial z} + 2\,\square i\,\square_d = 0 \qquad (13)$$

$$P_d = \square_d\,T_0 + \square_0\,T_d \qquad (14)$$

$$\frac{2\,\square i}{f}\square_0 q_d = -\frac{q_d}{Va} - \frac{\square^2}{\square M^2}\frac{\partial P_d}{\partial z} \qquad (15)$$

$$\square_0^* C_p\left[\frac{\square_0 q_d}{f}\frac{\partial T_0}{\partial z} - 2\,\square\, i\left(\frac{R}{C_p}P_d - \square_0 T_d\right)\right] = \frac{\square^* h}{r_h}(T_{md} - T_d) + \frac{\square^* k}{L_0^{*2}}\frac{\partial^2 T_d}{\partial z^2} \qquad (16)$$

$$(1-f)\,\square_m^* C_m[\,2\,\square\, i\, T_{md}] = \frac{f\,\square^* h}{r_h}(T_d - T_{md}) + \frac{(1-f)\,\square^* k_m}{L_0^{*2}}\frac{\partial^2 T_{md}}{\partial z^2} \qquad (17)$$

Equation (17) is easily solved if the last term, $k_m\partial^2 T_{md}/\partial z^2$, is neglected and if the oscillating matrix temperature is assumed to be proportional to the gas temperature oscillations, but with a phase shift. Then eq. 16 becomes (neglecting $k\,\partial^2 T_d/\partial z^2$):

$$\frac{\square_0 q_d}{f}\frac{\partial T_0}{\partial z} - 2\,\square\, i\left(\frac{R}{C_p}P_d - \square_0 T_d\right) = \frac{-2\,\square\, i\,\square}{1 + i\,\square}\,T_d, \qquad (18)$$

where $\square = \dfrac{2\,\square\,(1-f)\,\square_m^* C_m\, r_h}{f\,\square^* h}$ and $\square = \dfrac{(1-f)\,\square_m^* C_m}{f\,\square_0^* C_p}$.

The variables $\square_0$, P_0 and T_0 are determined by lower order equations and by static boundary conditions. The results are $P_0 = 1$, $T_0 = 1 + \square T\, z$ and $\square_0 = 1/T_0$, where $\square T$ is the temperature difference from the hot to the cold end of the regenerator. Equations (14), (15) and (18) can be used to express the three oscillating-term coefficients, q_d, T_d and $\square_d$ in terms of P_d and eq. (13) can use these expressions to arrive at the differential equation[2] for P_d:

$$0 = \frac{\partial^2 P_d}{\partial z^2} + \left[\frac{(1+\square\, i)\,\square\, T_0 + \square^2 + 1}{\left([\square\, T_0 + 1]^2 + \square^2\right) T_0} - \frac{\square\,(\square\, T_0 - i)}{\square^2\, T_0^{\,2} + 1}\right]\frac{\partial T_0}{\partial z}\frac{\partial P_d}{\partial z}$$
$$+\ \frac{4\,\square^2 M^2\,(\square\, i\, T_0 - 1)\left\{(\square - 1)[(1+\square\, i)\,\square\, T_0 + \square^2 + 1] - \square\left[(\square\, T_0 + 1)^2 + \square^2\right]\right\}P_d}{\square^2\left[(\square\, T_0 + 1)^2 + \square^2\right]T_0} \qquad (19)$$

With specified boundary conditions, eq. (19) can be solved for P_d by numerical methods and the result used to find q_d, T_d and $\square_d$. The boundary conditions will be provided by the requirements of matching pressures and mass flows at the two ends of the regenerator--at the compressor (hot end) and at the pulse tube (cold end).

There is one problem, however. When average enthalpy flow (time-averaged over one cycle) in the regenerator is calculated from these oscillating parameters, one finds:

$$\langle\dot h\rangle = C_p\,\langle\dot m\, T\rangle = \frac{C_p}{f}\,\langle\square\, q\, T\rangle = \frac{\square C_p}{2f}\,\square_0\, T_d\, q_d^{cc}$$

using the relationships $\square_0\, q_a = 0$ and $\square_a q_a + \square_0\, q_{2a} + \frac{\square_d}{2}\, q_d^{cc} = 0$ that come from the requirement that there is no average mass flow over a cycle. $\langle\,\rangle$ denotes a time average over one cycle and q^{cc} refers to the complex conjugate of q; $T_d\, q_d^{cc}/2$ is the zero frequency term arising from multiplying two fundamental–frequency expressions.

In general this is not constant with z in the regenerator. This means that there must be some conversion of enthalpy flux into heat that is conducted axially through the gas and/or conducted into the matrix and conducted axially through the matrix. For this to happen there must be a non-linear static temperature profile in the gas and/or the matrix. But T_0 is linear in z and it can be shown[2] that neither T_a nor T_{ma} can provide the necessary temperature profile correction. Therefore, the correction to the linear T_0 profile must come from T_{2a}.

Using higher order versions of the conservation equations, and taking the gas axial conduction term, $k\,\partial^2 T_{2a}/\partial z^2$, to be negligible, one can show that

$$\frac{\square_0^* C_p}{2}\frac{\partial\left(\square_0\, q_d^{cc}\, T_d\right)}{\partial z} = \frac{(1-f)\,\square^* k_m}{L_0^{*2}}\frac{\partial^2 T_{m2a}}{\partial z^2} \qquad (21)$$

which can be integrated to find T_{m2a} and

$$\frac{\square_0^* C_p}{2f}\frac{\partial\left(\square_0\, q_d^{cc}\, T_d\right)}{\partial z} = \frac{\square^* h}{r_h}\left(T_{m2a} - T_{2a}\right) \qquad (22)$$

which can be used to find T_{2a}.

We now have the corrected static temperature profile and the lowest order oscillating terms, P_d, T_d, $\square_d$ and q_d. In addition, we can calculate the average enthalpy flux and it obeys the second order energy conservation equations.

BOUNDARY CONDITIONS AT COMPRESSOR

In the compressor, using scaled parameters:

Pressure: $P_c = 1 + \square P_{cd}\, e^{i2\square t}$ (scaled by P_0^*) (23a)

Mass in the compressor: $m_c = 1 + \square m_{cd}\, e^{i2\square t}$ (23b)

(scaled by m_{ca}^*, the average mass in the compressor)

Distance above the piston: $\square_c = \square_{ca} + \square\square_{cd}\, e^{i2\square t}$ (scaled by L_0^*) (23c)

Temperature: $T_c = T_{ca} + \square T_{cd}\, e^{i2\square t}$ (scaled by T_0^*) (23d)

For an isothermal compressor, (essentially the same whether scaled or unscaled)

$$\frac{1}{P_c}\frac{\partial P_c}{\partial t} = -\frac{1}{\square_c}\frac{\partial \square_c}{\partial t} - \frac{1}{m_c}\frac{\partial m_c}{\partial t} \qquad (24)$$

But $P_c = P$ at entrance to regenerator ($z = 0$) and $\partial m_c/\partial t$ is mass flowing into regenerator at $z = 0$. Therefore $\partial m_c^*/\partial t = A_{regen}\, \square^*\, q^*$ in unscaled parameters. Applying scaling yields:

$$\partial m_c/\partial t = (F\,\square)\left(\square_0 + \square\square_d\, e^{2\square i t}\right)\left(q_d\, e^{2\square i t}\right) \qquad (25)$$

where $F = A_{regen}\, L_0^*\, \square_0^* / m_{ca}^*$.

Using eq. (25) and eq. (23c), keeping only the lowest order oscillating terms, eq. (24) becomes

$$2\square i\, \square P_d(z=0) = -\frac{2\square i\, \square\square_{cd}}{\square_{ca}} - F\,\square\square_0\, q_d \quad (\text{here, } \square_0 = 1) \qquad (26)$$

This equation provides one of the boundary conditions for the solution of eq. (19).

BOUNDARY CONDITIONS AT PULSE TUBE

In the pulse tube, using scaled parameters:

Pressure: $P_p = 1 + \square P_{pd}\, e^{i2\square t}$ (scaled by P_0^*) (27a)

Mass in the pulse tube from z to the warm end:

$$m_p = m_{pa} + m_{pd}\, e^{i2\square t} \quad (\text{scaled by } m_{ca}^*) \qquad (27b)$$

Temperature: $T_p = T_{pa} + \square T_{pd}\, e^{i2\square t}$ (scaled by T_0^*) (27c)

Based on the treatment of Radebaugh[3]:

$$\frac{\partial m_r^*}{\partial t^*} = \frac{A_{pt}\left(\square - L_{regen}\right)}{\square R\, T_{cold}^*}\frac{\partial P_p^*}{\partial t^*} + \frac{T_0^*}{T_{cold}^*}\frac{\partial m_o^*}{\partial t^*} \qquad (28)$$

where $\partial m_r^*/\partial t^*$ is the mass flow from the regenerator into the pulse tube at its cold end, P_p^* is the pressure in the pulse tube, $\partial m_o^*/\partial t^*$ is the mass flow through the orifice at the warm end, A_{pt} is the area of the pulse tube and $\square$ is the length of regenerator plus pulse tube. Eq. (28) scales to

$$\frac{\partial m_r}{\partial t} = G\frac{\partial P_p}{\partial t} + \frac{1}{T_{cold}}\frac{\partial m_o}{\partial t} \quad \text{where } G = A_{pt} L_{pt} \square_0^* T_0^* / \square m_{ca}^* T_{cold}^* \tag{29}$$

But $\frac{\partial m_r}{\partial t}$ is the mass flow from the regenerator at z = 1:

so
$$\frac{\partial m_r}{\partial t} = F \square\square_0(1)\, q_d(1) e^{2\square i t} \tag{30a}$$

and $\partial m_o/\partial t$ is the mass flow at the orifice (proportional to pressure):

$$\frac{\partial m_o}{\partial t} = k \square P_{pd}\, e^{2\square i t} = k \square P_d(1) e^{2\square i t} \tag{30b}$$

Therefore,
$$F \square\square_0(1)\, q_d(1) = \square\left(2\square i G + \frac{1}{T_{cold}} k\right) P_d(1) \tag{31}$$

This equation provides the second boundary condition for the solution of eq. (19).

At the present time, the solution of eq. (19) is carried out with temperature-independent values for thermal conductivity, viscosity and heat transfer coefficient. The effect of a temperature-dependent viscosity will be incorporated in the next improvement of the calculation. We are looking into ways to include the temperature dependence of the thermal conductivity and the temperature and velocity dependence of the heat transfer coefficient.

If all the dimensions and parameters of the system are inserted into the equations, the results for cooling power can be calculated. The enthalpy flow in the pulse tube is the fundamental cooling of the system from which various heat leaks must be subtracted. These are: axial conduction in the wall of the pulse tube, axial conduction in the wall and matrix of the regenerator, and enthalpy flux in the gas at the hot end of the regenerator. (This last term redistributes itself to become an enthalpy flux in the gas at the cold end of the regenerator plus an additional axial conduction through the wall and matrix as discussed above; the sum of these two contributions doesn't change, however, as it passes through the regenerator.) The orifice opening parameter is then varied to find a value that maximizes the net cooling power.

RESULTS FROM MODEL

When typical numbers are used for the various parameters in the above equations, it is noticed that the two terms in eq. (19) with denominators $(\square T_0 + 1)^2 + \square^2$ are very small, approximately 2 orders of magnitude less than the other terms in the equation. These terms come from the expression for the oscillating temperature. It would not distort the solution appreciably to ignore these terms in the calculation of the pressure. This is plausible since it would not be expected that the thermal behavior of the system would have a large effect on the mechanical behavior (pressure and velocity) of the system. Once the pressure and velocity were found from the simpler approach then the temperature oscillation could be found from the eq. (18). This is not the approach we take but it would certainly be reasonable if circumstances made it advantageous.

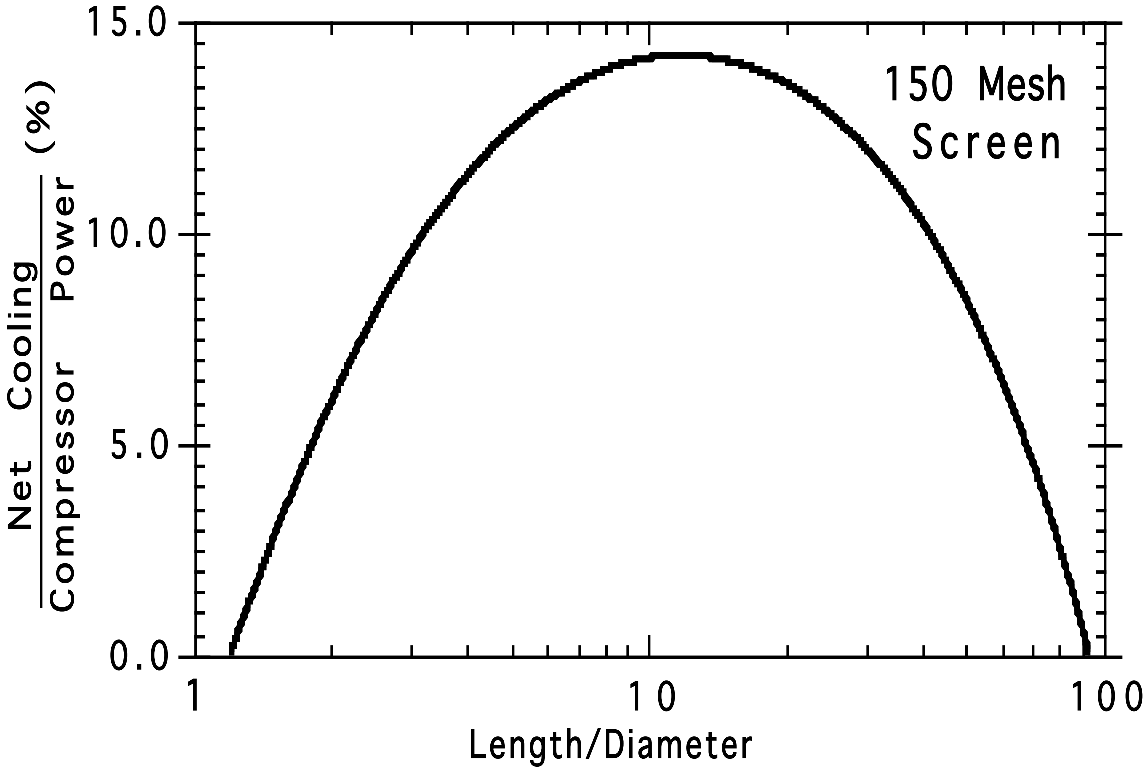

Fig. 1. Cooling power as a function of length-to-diameter ratio for 150-mesh screens.

It is interesting to ask how the performance of a pulse tube depends on the length-to-diameter ratio of the regenerator. For a constant volume of the regenerator, there should be an optimum length-to-diameter ratio since the axial-conduction heat leak will become very large at short lengths and the pressure drop will become very large at long lengths. We used our model to calculate some performance data for a typical orifice pulse tube operating between 300 K and 100 K using helium gas with γ=1.68. The cooling power we calculate is composed of the enthalpy flow in the pulse tube section reduced by the enthalpy flow through the regenerator toward the cold end and reduced by axial heat conduction through the walls of the pulse tube and regenerator. Figure 1 shows the results for a regenerator filled with 150-mesh screens. The optimum length/diameter ratio is 12 for this case.

When the regenerator screens are made finer by going to 250 mesh, the cooling power is higher at the optimum length/diameter ratio as shown in fig. 2. Because of the more-restrictive screens, the optimum length/diameter occurs at 3.6.

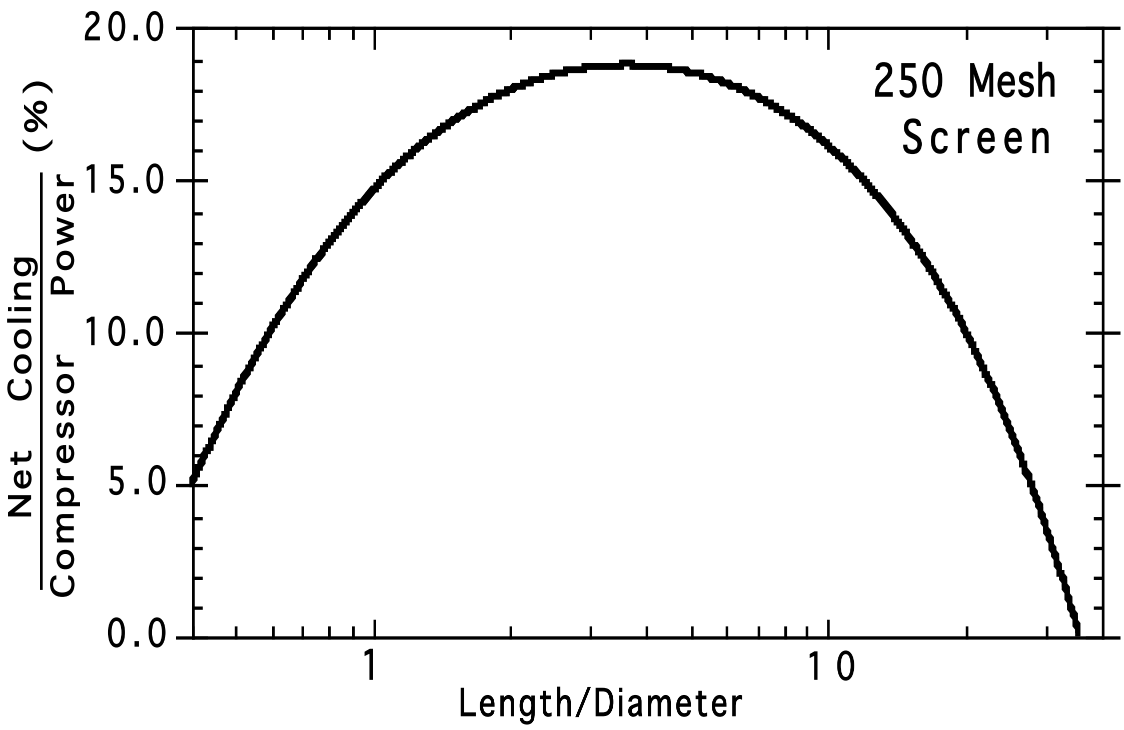

Fig. 2. Cooling power as a function of length-to-diameter ratio for 250-mesh screens.

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